

New coefficients for Hopf cyclic cohomology

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Abstract

In this note the categories of coefficients for Hopf cyclic cohomology of comodule algebras and comodule coalgebras are extended. We show that these new categories have two proper different subcategories where the smallest one is the known category of stable anti Yetter-Drinfeld modules. We prove that components of Hopf cyclic cohomology such as cup products work well with these new coefficients.

Introduction

Coefficients for Hopf cyclic cohomology was first introduced in [HKRS1] under the name of stable anti Yetter-Drinfeld (SAYD) modules for all four quantum symmetries i.e., (co)module (co)algebras. These coefficients are generalized in [HKR] for module algebras and module coalgebras. In the aforementioned reference a suitable class of coefficients for Hopf cyclic cohomology is introduced and is shown to be much larger than the category of SAYD modules. More precisely, it was shown that there are at least three noticeable categories of such coefficients by which the Hopf cyclic cohomology of module algebras and module coalgebras make sense. This new class of coefficients are indispensable as there are Hopf algebras which lack a large class of SAYD modules such as Connes-Moscovici Hopf algebra [RS]. In fact the new coefficients, on the contrary to the old ones, depend on both Hopf algebra and (co)algebra in question. By introducing several examples it is also shown that these extensions of coefficients are proper.

In this paper we complete [HKR] by introducing the new categories of coefficients for Hopf cyclic cohomology with respect to other two symmetries of

comodule algebras and comodule coalgebras which were remained open in [HKR]. One notes that Hopf cyclic cohomology of module coalgebras generalizes the Connes-Moscovici Hopf cyclic cohomology defined in [CM98]. The Hopf cyclic cohomology of module algebras generalizes the cyclic cohomology of algebras and twisted cyclic cohomology. In the case of comodule algebras one obtains the suitable coefficients for dual of Connes-Moscovici Hopf cyclic cohomology defined in [KR]. At the end we show that the components of Hopf cyclic cohomology such as cup products work well with these new coefficients.

Acknowledgement

The author would like to thank Bahram Rangipour for his valuable comments and discussions on this paper.

Notations: In this paper we denote a Hopf algebra by \mathcal{H} and the counit of a Hopf algebra by ε . We use the Sweedler summation notation $\Delta(h) = h^{(1)} \otimes h^{(2)}$ for the coproduct of a Hopf algebra. Furthermore $\blacktriangledown(h) = h_{<-1>} \otimes h_{<0>}$ and $\blacktriangledown(h) = h_{<0>} \otimes h_{<1>}$ are used for the left and right coactions of a coalgebra, respectively.

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1 Generalized Hopf cyclic cohomology with coefficients

In this section we introduce two categories of suitable coefficients for Hopf cyclic cohomology for comodule algebras and comodule coalgebras. Let us recall from [HKRS1] that a right-left SAYD module M over a Hopf algebra \mathcal{H} is a right module and a left comodule over \mathcal{H} satisfying the following

conditions.

$$\nabla(mh) = S(h^{(3)})m_{<-1>}h^{(1)} \otimes m_{<0>}h^{(2)}, \quad (\text{AYD condition}) \quad (1.1)$$

$$m_{<0>}m_{<-1>} = m, \quad (\text{Stability condition}). \quad (1.2)$$

1.1 The ${}^A\mathcal{H}$ -SAYD and HCC modules for comodule algebras

In this subsection we introduce a generalization of Hopf cyclic cohomology with coefficients with respect to the symmetry of a comodule algebra. Let M be a right-left SAYD module over \mathcal{H} and A be a left \mathcal{H} -comodule algebra. We let \mathcal{H} coacts on $A^{\otimes(n+1)}$ diagonally, *i.e.*

$$a_0 \otimes \cdots \otimes a_n \longmapsto a_{0<-1>} \cdots a_{n<-1>} \otimes a_{0<0>} \otimes \cdots \otimes a_{n<0>}.$$

Let $C^{\mathcal{H}}(A, M) := \text{Hom}^{\mathcal{H}}(A^{\otimes(n+1)}, M)$ denotes the set of all left \mathcal{H} -colinear morphisms. The following maps define a cocyclic module on $C^{\mathcal{H}}(A, M)$.

$$\begin{aligned} (\delta_i f)(a_0 \otimes \cdots \otimes a_n) &= f(a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n), \quad 0 \leq i < n, \\ (\delta_n f)(a_0 \otimes \cdots \otimes a_n) &= f(a_{n<0>} a_0 \otimes a_1 \cdots \otimes a_{n-1}) a_{n<-1>}, \\ (\sigma_i f)(a_0 \otimes \cdots \otimes a_n) &= f(a_0 \otimes \cdots \otimes a_i \otimes 1 \otimes \cdots \otimes a_n), \quad 0 \leq i < n, \\ (\tau_n f)(a_0 \otimes \cdots \otimes a_n) &= f(a_{n<0>} \otimes a_0 \cdots \otimes a_{n-1}) a_{n<-1>}. \end{aligned} \quad (1.3)$$

Definition 1.1. *Let A be a left \mathcal{H} -comodule algebra. A right-left module comodule M over \mathcal{H} is called an ${}^A\mathcal{H}$ -SAYD module if for all $m \in M$, $h \in \mathcal{H}$, $a \in A$ and $\varphi \in \text{Hom}^{\mathcal{H}}(A^{\otimes(n+1)}, M)$ the following ${}^A\mathcal{H}$ -AYD and stability conditions are satisfied.*

i)

$$\begin{aligned} & \left(\varphi(a_{<0>} \otimes \tilde{b}) a_{<-1>} \right)_{<-1>} \otimes \left(\varphi(a_{<0>} \otimes \tilde{b}) a_{<-1>} \right)_{<0>} = \\ & S(a_{<-1>}^{(3)}) \varphi(a_{<0>} \otimes \tilde{b})_{<-1>} a_{<-1>}^{(1)} \otimes \varphi(a_{<0>} \otimes \tilde{b})_{<0>} a_{<-1>}^{(2)}, \end{aligned}$$

where $\tilde{b} = a_1 \otimes \cdots \otimes a_n$.

ii) $\varphi(\tilde{a}_{<0>}) \tilde{a}_{<-1>} = \varphi(\tilde{a})$.

Remark 1.2. Obviously for any \mathcal{H} -comodule algebra A , any SAYD module over \mathcal{H} is a ${}^A\mathcal{H}$ -SAYD module.

The following lemma generalizes the notion of modular pair in involution [CM98].

Lemma 1.3. *Let A be a left \mathcal{H} -comodule algebra, δ be a character and be σ a group like element for \mathcal{H} . If (σ, δ) be a modular pair, i.e. $\delta(\sigma) = 1$, and in ${}^A\mathcal{H}$ -in involution, i.e.*

$$\sigma^{-1}S_\delta^2(a_{<-1>})\sigma \otimes a_{<0>} = a_{<-1>} \otimes a_{<0>}, \quad (1.4)$$

for all $h \in \mathcal{H}, a \in A$, then ${}^\sigma\mathbb{C}_\delta$ is a ${}^A\mathcal{H}$ -SAYD module.

Proof. For any $\varphi \in \text{Hom}^{\mathcal{H}}(A^{\otimes(n+1)}, \mathbb{C})$ the following computation proves the ${}^A\mathcal{H}$ -AYD condition.

$$\begin{aligned} & \left(\delta(a_{<-1>})\varphi(a_{<0>} \otimes \tilde{b}) \right)_{<-1>} \otimes \left(\delta(a_{<-1>})\varphi(a_{<0>} \otimes \tilde{b}) \right)_{<0>} \\ &= \delta(a_{<-1>})\varphi(a_{<0>} \otimes \tilde{b})\sigma \otimes 1 \\ &= \delta(a_{<-1>}^{(3)})\varphi(a_{<0>} \otimes \tilde{b})\sigma S^{-1}(a_{<-1>}^{(2)})a_{<-1>}^{(1)} \otimes 1 \\ &= \varphi(a_{<0>} \otimes \tilde{b})\sigma S_\delta^{-1}(a_{<-1>}^{(2)})\sigma^{-1}\sigma a_{<-1>}^{(1)} \otimes 1 \\ &= \varphi(a_{<0>} \otimes \tilde{b})S_\delta^{-1}(\sigma a_{<-1>}^{(2)}\sigma^{-1})\sigma a_{<-1>}^{(1)} \otimes 1 \\ &= \varphi(a_{<0>} \otimes \tilde{b})S_\delta^{-1}(S_\delta^2(a_{<0>} \otimes a_{<-1>}))\sigma a_{<-1>} \otimes 1 \\ &= \varphi(a_{<0>} \otimes \tilde{b})S_\delta^{-1}(S_\delta^2(a_{<-1>}^{(2)}))\sigma a_{<-1>}^{(1)} \otimes 1 \\ &= \varphi(a_{<0>} \otimes \tilde{b})S(a_{<-1>}^{(3)})\delta(a_{<-1>}^{(2)})\sigma a_{<-1>}^{(1)} \otimes 1 \\ &= S(a_{<-1>}^{(3)})\varphi(a_{<0>} \otimes \tilde{b})_{<-1>} a_{<-1>}^{(1)} \otimes \varphi(a_{<0>} \otimes \tilde{b})_{<0>} a_{<-1>}^{(2)}. \end{aligned}$$

We use anti-algebra map property of S_δ^{-1} in the fourth equality, the coassociativity and ${}^A\mathcal{H}$ -in involution condition for the element $s_{<0>} \in A$ in fifth equality. One has the ${}^A\mathcal{H}$ -stability condition by the modular pair condition. \square

Proposition 1.4. *Let (δ, σ) be a modular pair for \mathcal{H} and A be a left \mathcal{H} -comodule algebra. We define the following subspace of A ,*

$$B = \{a \in A \mid \sigma^{-1}S_\delta^2(a_{<-1>})\sigma \otimes a_{<0>} = a_{<-1>} \otimes a_{<0>}\}.$$

Then B is a \mathcal{H} -comodule subalgebra of A . Furthermore ${}^\sigma\mathbb{C}_\delta$ is a ${}^B\mathcal{H}$ -SAYD module.

Proof. Since A is a \mathcal{H} -comodule algebra and S_δ^2 is an algebra map, the following computation shows that B is a subalgebra of A . More precisely

for any $a, b \in B$ we have

$$\begin{aligned} \sigma^{-1}S_{\delta}^2((ab)_{<-1>})\sigma \otimes (ab)_{<0>} &= \sigma^{-1}S_{\delta}^2(a_{<-1>})S_{\delta}^2(b_{<-1>})\sigma \otimes a_{<0>}b_{<0>} \\ &= \sigma^{-1}S_{\delta}^2(a_{<-1>})\sigma\sigma^{-1}S_{\delta}^2(b_{<-1>})\sigma \otimes a_{<0>}b_{<0>} = a_{<-1>}b_{<-1>} \otimes a_{<0>}b_{<0>}. \end{aligned}$$

To prove that \mathcal{H} coacts on B , it is enough to show that for any $b \in B$ we have $b_{<-1>} \otimes b_{<0>} \in \mathcal{H} \otimes B$.

$$\begin{aligned} &b_{<-1>} \otimes \sigma^{-1}S_{\delta}^2(b_{<0><-1>})\sigma \otimes b_{<0><0>} \\ &= \sigma^{-1}S_{\delta}^2(b_{<-1>})\sigma \otimes \sigma^{-1}S_{\delta}^2(b_{<0><-1>})\sigma \otimes b_{<0><0>} \\ &= \sigma^{-1}S_{\delta}^2(b_{<-1>}^{(1)})\sigma \otimes \sigma^{-1}S_{\delta}^2(b_{<-1>}^{(2)})\sigma \otimes b_{<0>} \\ &= \sigma^{-1}S_{\delta}^2(b_{<-1>}^{(1)})\sigma \otimes \sigma^{-1}S_{\delta}^2(b_{<-1>}^{(2)})\sigma \otimes b_{<0>} \\ &= (\sigma^{-1}S_{\delta}^2(b_{<-1>})\sigma)^{(1)} \otimes (\sigma^{-1}S_{\delta}^2(b_{<-1>})\sigma)^{(2)} \otimes b_{<0>} \\ &= b_{<-1>}^{(1)} \otimes b_{<-1>}^{(2)} \otimes b_{<0>} = b_{<-1>} \otimes b_{<0><-1>} \otimes b_{<0><0>}. \end{aligned}$$

We use $b \in B$ in the first equality, the coassociativity of the coaction in the second equality and the coalgebra map property of the map S_{δ}^2 in the third equality. \square

Definition 1.5. Let \mathcal{H} be a Hopf algebra and A be an algebra which is also an \mathcal{H} -comodule where the coaction of \mathcal{H} on $A^{\otimes(n+1)}$ is diagonal. A module-comodule M over \mathcal{H} is called an ${}^A\mathcal{H}$ -Hopf cyclic coefficients, and abbreviated by ${}^A\mathcal{H}$ -HCC, if the cosimplicial and cyclic operators on $\text{Hom}^H(A^{\otimes(n+1)}, M)$ are well-defined and make it a cocyclic module.

Proposition 1.6. Let M be a right-left module comodule over \mathcal{H} and A a right \mathcal{H} -comodule algebra. If M is a ${}^A\mathcal{H}$ -SAYD module, then M is an ${}^A\mathcal{H}$ -HCC.

Proof. It is enough to show that the cyclic map is well-defined. The following

computation proves that τf is a left \mathcal{H} -comodule map.

$$\begin{aligned}
& ((\tau_n f)(a_0 \otimes \cdots \otimes a_n))_{<-1>} \otimes ((\tau_n f)(a_0 \otimes \cdots \otimes a_n))_{<0>} \\
&= (f(a_{n<0>} \otimes a_0 \otimes \cdots \otimes a_{n-1})a_{n<-1>})_{<-1>} \otimes \\
& (f(a_{n<0>} \otimes a_0 \otimes \cdots \otimes a_{n-1})a_{n<-1>})_{<0>} \\
&= S(a_{n<-1>}^{(3)})(f(a_{n<0>} \otimes a_0 \otimes \cdots \otimes a_{n-1}))_{<-1>} a_{n<-1>}^{(2)} \otimes \\
& \otimes (f(a_{n<0>} \otimes a_0 \otimes \cdots \otimes a_{n-1}))_{<0>} a_{n<-1>}^{(2)} \\
&= S(a_{n<-1>}^{(3)})a_{n<0><-1>} a_{0<-1>} \cdots a_{n-1<-1>} a_{n<-1>}^{(1)} \otimes \\
& \otimes f(a_{n<0><0>} \otimes a_{0<0>} \otimes \cdots \otimes a_{n-1<0>})a_{n<-1>}^{(2)} \\
&= S(a_{n<-1>}^{(3)})a_{n<-1>}^{(4)} a_{0<-1>} \cdots a_{n-1<-1>} a_{n<-1>}^{(1)} \otimes \\
& \otimes f(a_{n<0>} \otimes a_{0<0>} \otimes \cdots \otimes a_{n-1<0>})a_{n<-1>}^{(2)} \\
&= a_{0<-1>} \cdots a_{n<-1>} \otimes f(a_{n<0><0>} \otimes a_{0<0>} \otimes \cdots \otimes a_{n-1<0>})a_{n<0><-1>} \\
&= a_{0<-1>} \cdots a_{n<-1>} \otimes (\tau f)(a_{0<0>} \otimes \cdots \otimes a_{n<0>}).
\end{aligned}$$

We use ${}^A\mathcal{H}$ -AYD condition in the second equality, the \mathcal{H} -comodule map property of f in the third equality and the coassociativity of the coaction for the element a_n in the fourth and fifth equalities. For cyclicity, using ${}_A\mathcal{H}$ -stability condition we have

$$\begin{aligned}
\tau^{n+1}(a_0 \otimes \cdots \otimes a_n) &= f(a_{0<0>} \otimes \cdots \otimes a_{n<0>})a_{0<-1>} \cdots a_{n<-1>} \\
&= f(\tilde{a}_{<0>})\tilde{a}_{<-1>} = f(a_0 \otimes \cdots \otimes a_n).
\end{aligned}$$

□

Lemma 1.7. *Let A be a left \mathcal{H} -comodule algebra. If \mathcal{H} coacts on A commutatively, i.e. for any $\tilde{a} \in A^{\otimes(n+1)}$ we have*

$$\tilde{a}_{<0>} \otimes \tilde{a}_{<-1>} g = \tilde{a}_{<0>} \otimes g \tilde{a}_{<-1>}, \quad g \in \mathcal{H}, a \in A, \quad (1.5)$$

then any comodule M over \mathcal{H} with a trivial action defines a ${}^A\mathcal{H}$ -SAYD module.

Proof. The following computation proves the ${}^A\mathcal{H}$ -AYD condition.

$$\begin{aligned}
f(\tilde{a}h)_{<-1>} \otimes f(\tilde{a}h)_{<-1>} &= \varepsilon(h)f(\tilde{a})_{<-1>} \otimes f(\tilde{a})_{<0>} \\
&= \varepsilon(h)a_{<-1>} \otimes f(\tilde{a}_{<0>}) = S(h^{(2)})h^{(1)}\tilde{a}_{<-1>} \otimes f(\tilde{a}_{<0>}) \\
&= S(h^{(3)})\tilde{a}_{<-1>} h^{(1)} \otimes \varepsilon(h^{(2)})f(\tilde{a}_{<0>}) \\
&= S(h^{(3)})f(\tilde{a})_{<-1>} h^{(1)} \otimes f(\tilde{a})_{<0>} h^{(2)}.
\end{aligned}$$

We use (1.5) in the fourth equality. The counitality of the coaction implies the ${}^A\mathcal{H}$ -stability condition. \square

One notes that for any bicrossed product Hopf algebra $\mathcal{H} = \mathcal{F} \blacktriangleright \mathcal{U}$, the Hopf algebra \mathcal{F} is a right \mathcal{H} -comodule algebra by the action defined by

$$f \longmapsto f^{(1)} \otimes (f^{(2)} \blacktriangleright 1_{\mathcal{U}}). \quad (1.6)$$

Here we introduce an example of a commutative coaction.

Example 1.8. Let $\mathcal{H} = \mathcal{F} \blacktriangleright \mathcal{U}$ be a bicrossed Hopf algebra where \mathcal{U} is not commutative. Suppose \mathcal{F} is commutative and \mathcal{U} acts trivially on \mathcal{F} . With respect to the coaction defined in (1.6), \mathcal{H} coacts commutatively on \mathcal{H} -comodule algebra \mathcal{F} .

$$\begin{aligned} & \tilde{f}_{<0>} \otimes \tilde{f}_{<1>} (g \blacktriangleright u) \\ &= f_1^{(1)} \otimes \cdots \otimes f_n^{(1)} \otimes (f_1^{(2)} \blacktriangleright 1) \cdots (f_n^{(2)} \blacktriangleright 1) (g \blacktriangleright u) \\ &= f_1^{(1)} \otimes \cdots \otimes f_n^{(1)} \otimes (f_1^{(2)} \cdots f_n^{(2)} g \blacktriangleright u) \\ &= f_1^{(1)} \otimes \cdots \otimes f_n^{(1)} \otimes (g f_1^{(2)} \cdots f_n^{(2)} \blacktriangleright u) \\ &= f_1^{(1)} \otimes \cdots \otimes f_n^{(1)} \otimes (g u^{(1)} \triangleright (f_1^{(2)} \cdots f_n^{(2)}) \blacktriangleright u^{(2)}) \\ &= f_1^{(1)} \otimes \cdots \otimes f_n^{(1)} \otimes (g \blacktriangleright u) (f_1^{(2)} \blacktriangleright 1) \cdots (f_n^{(2)} \blacktriangleright 1) \\ &= \tilde{f}_{<0>} \otimes (g \blacktriangleright u) \tilde{f}_{<1>}. \end{aligned}$$

We use the commutativity of \mathcal{F} in the third equality.

Lemma 1.9. Let A be a left \mathcal{H} -comodule algebra. If \mathcal{H} coacts on A cocommutatively, i.e. for any $a \in A$, $\tilde{b} \in A^{\otimes n}$ and $h \in \mathcal{H}$;

$$\tilde{b}_{<-1>} a_{<-1>}^{(1)} \otimes a_{<-1>}^{(2)} \otimes a_{<0>} \otimes \tilde{b}_{<0>} = a_{<-1>}^{(2)} \tilde{b}_{<-1>} \otimes a_{<-1>}^{(1)} \otimes a_{<0>} \otimes \tilde{b}_{<0>}. \quad (1.7)$$

Then any module M over \mathcal{H} with the trivial coaction defines a ${}^A\mathcal{H}$ -HCC module.

Proof. It is enough to show that the cyclic map is a left \mathcal{H} -comodule map.

$$\begin{aligned}
& (a_0 \otimes \cdots \otimes a_n)_{<-1>} \otimes (\tau_n f)((a_0 \otimes \cdots \otimes a_n)_{<0>}) \\
&= a_{0<-1>} \cdots a_{n<-1>} \otimes (\tau_n f)(a_{0<0>} \otimes \cdots \otimes a_{n<0>}) \\
&= a_{0<-1>} \cdots a_{n<-1>} \otimes f(a_{n<0><0>} \otimes a_{0<0>} \cdots \otimes a_{n-1<0>}) a_{n<0><-1>} \\
&= a_{0<-1>} \cdots a_{n-1<-1>} a_{n<-1>}^{(1)} \otimes f(a_{n<0>} \otimes a_{0<0>} \cdots \otimes a_{n-1<0>}) a_{n<-1>}^{(2)} \\
&= a_{n<-1>}^{(2)} a_{0<-1>} \cdots a_{n-1<-1>} \otimes f(a_{n<0>} \otimes a_{0<0>} \cdots \otimes a_{n-1<0>}) a_{n<-1>}^{(1)} \\
&= f(a_{n<0>} \otimes a_0 \otimes \cdots \otimes a_{n-1})_{<-1>} \otimes f(a_{n<0>} \otimes a_0 \otimes \cdots \otimes a_{n-1})_{<0>} a_{n<-1>} \\
&= 1 \otimes f(a_{n<0>} \otimes a_0 \otimes \cdots \otimes a_{n-1}) a_{n<-1>} \\
&= (f(a_{n<0>} \otimes a_0 \otimes \cdots \otimes a_{n-1}) a_{n<-1>})_{<-1>} \otimes \\
&\quad \otimes (f(a_{n<0>} \otimes a_0 \otimes \cdots \otimes a_{n-1}) a_{n<-1>})_{<0>} \\
&= ((\tau_n f)(a_0 \otimes \cdots \otimes a_n))_{<-1>} \otimes ((\tau_n f)(a_0 \otimes \cdots \otimes a_n))_{<0>}.
\end{aligned}$$

We use the relation (1.7) in the fourth equality, the coassociativity of the coaction for the element a_n and the left \mathcal{H} -comodule map property of the map f in the fifth equality and the triviality of the coaction of \mathcal{H} on M in the sixth and seventh equalities. \square

Here we introduce an example of a cocommutative coaction.

Example 1.10. Let $\mathcal{H} = \mathcal{F} \blacktriangleleft \mathcal{U}$ be a bicrossed product Hopf algebra where F is commutative and cocommutative. Then we consider the right \mathcal{H} -comodule algebra \mathcal{F} with the coaction defined in (1.6). Then the following computation shows that this coaction is cocommutative.

$$\begin{aligned}
& g_{<0>} \otimes \tilde{f}_{<0>} \otimes \tilde{f}_{<1>} g_{<1>}^{(1)} \otimes g_{<1>}^{(2)} \\
&= g_{<0>} \otimes f_{1<0>} \otimes \cdots \otimes f_{n<0>} \otimes f_{1<1>} \cdots f_{n<1>} g_{<1>}^{(1)} \otimes g_{<1>}^{(2)} \\
&= g^{(1)} \otimes f_1^{(1)} \otimes \cdots \otimes f_n^{(1)} \otimes f_1^{(2)} \cdots f_n^{(2)} g^{(2)} \otimes g^{(3)} \\
&= g^{(1)} \otimes f_1^{(1)} \otimes \cdots \otimes f_n^{(1)} \otimes g^{(3)} f_1^{(2)} \cdots f_n^{(2)} \otimes g^{(2)} \\
&= g_{<0>} \otimes f_{1<0>} \otimes \cdots \otimes f_{n<0>} \otimes g_{<1>}^{(2)} f_{1<1>} \cdots f_{n<1>} \otimes g_{<1>}^{(1)} \\
&= g_{<0>} \otimes \tilde{f}_{<0>} \otimes g_{<1>}^{(2)} \tilde{f}_{<1>} \otimes g_{<1>}^{(1)}.
\end{aligned}$$

We use the commutativity and cocommutativity of \mathcal{F} in the third equality. Therefore any module M over $\mathcal{H} = \mathcal{F} \blacktriangleleft \mathcal{U}$ with the trivial coaction defines a $\mathcal{F}\mathcal{H}$ -HCC. One easily checks that since \mathcal{F} is cocommutative any module M with the trivial coaction in fact is a $\mathcal{F}\mathcal{H}$ -SAYD module. Therefore the $\mathcal{F}\mathcal{H}$ -HCC property of M is not a result of the cocommutativity of the coaction.

Example 1.11. In the special case of the previous example, consider \mathcal{H} to be the co-opposite Hopf algebra of Schwarzian Hopf algebra \mathcal{H}_{1s}^{cop} which is in fact the quotient of co-opposite Hopf algebra of Connes-Moscovici Hopf algebra H_1^{cop} by the ideal S generated by the Schwarzian element $\delta'_2 = \delta_2 - \frac{1}{2}\delta_1^2$. In fact \mathcal{H}_{1s}^{cop} is generated by X , Y and $Z = \delta_1$ where

$$[Y, X] = X, \quad [Y, Z] = Z, \quad [X, Z] = \frac{1}{2}Z^2.$$

The coalgebra structure and antipode are defined similar to the one for \mathcal{H}_1^{cop} . The Hopf algebra \mathcal{U} acts on \mathcal{F} via

$$X \triangleright Z = -\frac{1}{2}Z^2, \quad Y \triangleright Z = -Z,$$

and \mathcal{F} coacts on \mathcal{U} via

$$\blacktriangledown(X) = X \otimes 1 + Y \otimes Z, \quad \blacktriangledown(Y) = Y \otimes 1.$$

Indeed we have $\mathcal{H}_1^{cop} = \mathbb{C}[Z] \blacktriangleright \mathcal{U}$.

Remark 1.12. If $\mathcal{SAYD}_{\mathcal{H}}$ denotes the category of SAYD modules, ${}^A\mathcal{H}\text{-}\mathcal{SAYD}$ denotes the category of ${}^A\mathcal{H}$ -SAYD modules and ${}^A\mathcal{H}\text{-}\mathcal{HCC}$ denotes the category of ${}^A\mathcal{H}$ -Hopf cyclic cohomology coefficients, then one has the following proper inclusions of categories.

$$\mathcal{SAYD}_{\mathcal{H}} \subsetneq {}^A\mathcal{H}\text{-}\mathcal{SAYD} \subsetneq {}^A\mathcal{H}\text{-}\mathcal{HCC}.$$

1.2 The \mathcal{H}^C -SAYD and HCC modules for comodule coalgebras

In this subsection we generalize the Hopf cyclic cohomology of comodule coalgebras with coefficients. Let C be a right \mathcal{H} -comodule coalgebra and M a right-left SAYD module on \mathcal{H} . We set

$$\mathcal{H}C^n(C, M) = C^{\otimes(n+1)} \square_{\mathcal{H}} M. \quad (1.8)$$

The following maps define a cocyclic module for $\mathcal{H}C^n(C, M)$ [HR2].

$$\begin{aligned} \delta_i(c_0 \otimes \cdots \otimes c_n \otimes m) &= c_0 \otimes \cdots \otimes \Delta(c_i) \otimes \cdots \otimes c_n \otimes m, \\ \delta_n(c_0 \otimes \cdots \otimes c_n \otimes m) &= c_0^{(2)} \otimes c_1 \otimes \cdots \otimes c_n \otimes c_0^{(1)}_{<0>} \otimes m \triangleleft c_0^{(1)}_{<1>}, \\ \sigma_i(c_0 \otimes \cdots \otimes c_n \otimes m) &= c_0 \otimes \cdots \otimes \varepsilon(c_{i+1}) \otimes \cdots \otimes c_n \otimes m, \\ \tau_n(c_0 \otimes \cdots \otimes c_n \otimes m) &= c_1 \otimes \cdots \otimes c_n \otimes c_{0<0>} \otimes m \triangleleft c_{0<1>}. \end{aligned} \quad (1.9)$$

Here $C^{\otimes(n+1)}$ is a right \mathcal{H} -comodule by diagonal coaction.

Definition 1.13. Let \mathcal{H} be a Hopf algebra and C be a right \mathcal{H} -comodule coalgebra. A right-left module-comodule M over \mathcal{H} is called an \mathcal{H}^C -SAYD module if the following \mathcal{H}^C -AYD and stability conditions are satisfied.

$$i) \quad c_{<0>} \otimes (mc_{<1>})_{<-1>} \otimes (mc_{<1>})_{<0>} = \\ c_{<0>} \otimes S(c_{<1>}^{(3)})m_{<-1>}c_{<1>}^{(1)} \otimes m_{<0>}c_{<1>}^{(2)},$$

$$ii) \quad \tilde{d}_{<0>} \square_{\mathcal{H}} m \tilde{d}_{<1>} = \tilde{d} \square_{\mathcal{H}} m,$$

where $\tilde{d} = c_0 \otimes \cdots \otimes c_n$.

One notices that since $\tilde{d} \otimes m \in C^{\otimes(n+1)} \square_{\mathcal{H}} M$, then stability condition is equivalent to

$$\tilde{d} \square_{\mathcal{H}} m_{<0>} m_{<-1>} = \tilde{d} \square_{\mathcal{H}} m.$$

Remark 1.14. Any SAYD module over \mathcal{H} implies a \mathcal{H}^C -SAYD module.

Lemma 1.15. Let C be a right \mathcal{H} -comodule coalgebra. If (σ, δ) be a modular pair and \mathcal{H}^C -in involution, i.e.

$$c_{<0>} \otimes S_{\delta}^2(c_{<1>}) = c_{<0>} \otimes \sigma c_{<1>} \sigma^{-1}, \quad c \in C, \quad (1.10)$$

then $\sigma \mathbb{C}_{\delta}$ is a \mathcal{H}^C -SAYD module.

Proof. The following computation proves the \mathcal{H}^C -AYD condition.

$$\begin{aligned} & c_{<0>} \otimes \sigma \delta(c_{<1>}) \otimes 1 \\ &= c_{<0>} \otimes \sigma S^{-1}(c_{<1>}^{(2)}) \delta(c_{<1>}^{(3)}) c_{<1>}^{(1)} \otimes 1 \\ &= c_{<0>} \otimes \sigma S_{\delta}^{-1}(c_{<1>}^{(2)}) \sigma^{-1} \sigma c_{<1>}^{(1)} \otimes 1 \\ &= c_{<0>} \otimes S_{\delta}^{-1}(\sigma c_{<1>}^{(2)} \sigma^{-1}) \sigma c_{<1>}^{(1)} \otimes 1 \\ &= c_{<0>} \otimes S_{\delta}^{-1}(S_{\delta}^2(c_{<1>})) \sigma c_{<1>} \otimes 1 \\ &= c_{<0>} \otimes S_{\delta}(c_{<1>}^{(2)}) \sigma c_{<1>}^{(1)} \otimes 1 \\ &= c_{<0>} \otimes S(c_{<1>}^{(3)}) \sigma c_{<1>}^{(1)} \otimes \delta(c_{<1>}^{(2)}). \end{aligned}$$

We use the coassociativity of the coaction and \mathcal{H}^C -in involution property in the fourth equality. The stability condition is obvious by the modular pair property. \square

Lemma 1.16. If \mathcal{H} coacts on C commutatively, i.e.

$$c_{<0>} \otimes hc_{<1>} = c_{<0>} \otimes c_{<1>} h \quad (1.11)$$

then any \mathcal{H} -comodule M with trivial action becomes a \mathcal{H}^C -SAYD module.

Proof. The \mathcal{H}^C -stability condition is obvious. The following computation proves the \mathcal{H}^C -AYD condition.

$$\begin{aligned}
& c_{<0>} \varepsilon(c_{<1>}) \otimes m_{<-1>} \otimes m_{<0>} \\
&= c_{<0>} \otimes S(c_{<1>}^{(2)}) c_{<1>}^{(1)} m_{<-1>} \otimes m_{<0>} \\
&= c_{<0>} \otimes S(c_{<1>}) c_{<0>} c_{<1>} m_{<-1>} \otimes m_{<0>} \\
&= c_{<0>} \otimes S(c_{<1>}) m_{<-1>} c_{<0>} c_{<1>} \otimes m_{<0>} \\
&= c_{<0>} \otimes S(c_{<1>}^{(3)}) m_{<-1>} c_{<1>}^{(1)} \otimes m_{<0>} \varepsilon(c_{<1>}^{(2)}) \\
&= c_{<0>} \otimes S(c_{<1>}^{(3)}) m_{<-1>} c_{<1>}^{(1)} \otimes m_{<0>} c_{<1>}^{(2)}.
\end{aligned}$$

We use the (1.11) in the third equality and coassociativity of the coaction in the fourth equality. \square

Let $\mathcal{H} = \mathcal{F} \blacktriangleright \mathcal{U}$ be any bicrossed product Hopf algebra where $u \mapsto u_{<0>} \otimes u_{<1>}$ denotes the coaction of \mathcal{F} on \mathcal{U} . The Hopf algebra \mathcal{U} is a \mathcal{H} -comodule coalgebra by the following coaction.

$$u \mapsto u_{<0>} \otimes u_{<1>} \otimes 1_{\mathcal{U}}. \quad (1.12)$$

Example 1.17. Let $\mathcal{H} = \mathcal{F} \blacktriangleright \mathcal{U}$ be a bicrossed product Hopf algebra. If \mathcal{F} is commutative and \mathcal{U} acts trivially on \mathcal{F} , then \mathcal{H} coacts commutatively on \mathcal{U} .

Definition 1.18. Let \mathcal{H} be a Hopf algebra and C be a coalgebra, which is also a \mathcal{H} -comodule where the coaction of \mathcal{H} on $C^{\otimes(n+1)}$ is diagonal. A module-comodule M over \mathcal{H} is called a \mathcal{H}^C -Hopf cyclic coefficients and abbreviated by \mathcal{H}^C -HCC, if the cosimplicial and cyclic operators on $C^{\otimes(n+1)} \square_{\mathcal{H}} M$ are well-defined and make it a cocyclic module.

Proposition 1.19. Let M be a right-left module-comodule over \mathcal{H} and C be a right \mathcal{H} -comodule coalgebra. If M is a \mathcal{H}^C -SAYD module then M is an \mathcal{H}^C -HCC.

Proof. It is enough to show that the cyclic map τ is well-defined. The following computation shows that $\tau_n(\tilde{c} \otimes m) \in {}^{\mathcal{H}}C^n(C, M)$, where $\tilde{c} =$

$$c_0 \otimes \cdots \otimes c_n.$$

$$\begin{aligned}
& c_1 \otimes \cdots \otimes c_n \otimes c_{0<0>} \otimes (mc_{0<1>})_{<-1>} \otimes (mc_{0<1>})_{<0>} \\
&= c_1 \otimes \cdots \otimes c_n \otimes c_{0<0>} \otimes S(c_{0<1>})^{(3)} m_{<-1>} c_{0<1>}^{(1)} \otimes m_{<0>} c_{0<1>}^{(2)} \\
&= c_{1<0>} \otimes \cdots \otimes c_{n<0>} \otimes c_{0<0><0>} \otimes S(c_{0<0><1>})^{(3)} c_{0<1>} \cdots c_{n<1>} c_{0<0><1>}^{(1)} \otimes \\
&\quad \otimes m_{<0>} c_{0<0><1>}^{(2)} \\
&= c_{1<0>} \otimes \cdots \otimes c_{n<0>} \otimes c_{0<0>} \otimes S(c_{0<1>})^{(3)} c_{0<1>}^{(4)} c_{1<1>} \cdots c_{n<1>} c_{0<1>}^{(1)} \otimes \\
&\quad \otimes m_{<0>} c_{0<1>}^{(2)} \\
&= c_{1<0>} \otimes \cdots \otimes c_{n<0>} \otimes c_{0<0>} \otimes c_{1<1>} \cdots c_{n<1>} c_{0<1>}^{(1)} \otimes m_{<0>} c_{0<1>}^{(2)} \\
&= c_{1<0>} \otimes \cdots \otimes c_{n<0>} \otimes c_{0<0><0>} \otimes c_{1<1>} \cdots c_{n<1>} c_{0<0><1>} \otimes m_{<0>} c_{0<1>}.
\end{aligned}$$

We use the \mathcal{H}^C -AYD condition in the first equality, the relation $c_0 \otimes \cdots \otimes c_n \otimes m \in C^{\otimes(n+1)} \square_{\mathcal{H}} M$ in the second equality, the coassociativity of the coaction for the element a_0 in the third and last equalities. To prove the cyclicity, using the \mathcal{H}^C -stability condition we have,

$$\begin{aligned}
\tau_n^{n+1}(c_0 \otimes \cdots \otimes c_n \otimes m) &= c_{0<0>} \otimes \cdots \otimes c_{n<0>} \otimes mc_{0<1>} \cdots c_{n<1>} \\
&= c_0 \otimes \cdots \otimes c_n \otimes m_{<0>} m_{<-1>} = c_0 \otimes \cdots \otimes c_n \otimes m.
\end{aligned}$$

□

Lemma 1.20. *Let C be a \mathcal{H} -comodule coalgebra. If the coaction of \mathcal{H} on C is cocommutative, i.e.*

$$\tilde{c}_{<0>} \otimes d_{<0>} \otimes \tilde{c}_{<1>} d_{<1>}^{(1)} \otimes d_{<1>}^{(2)} = \tilde{c}_{<0>} \otimes d_{<0>} \otimes d_{<1>}^{(2)} \tilde{c}_{<1>} \otimes d_{<1>}^{(1)}, \quad (1.13)$$

for all $c \in C$ and $h \in \mathcal{H}$, then any module M over \mathcal{H} with the trivial coaction defines a \mathcal{H}^C -HCC.

Proof. It is enough to show that the cyclic map is well-defined. The following computation proves $\tau(\tilde{c} \otimes m) \in C^{\otimes(n+1)} \square_H M$.

$$\begin{aligned}
& c_{1<0>} \otimes \cdots \otimes c_{n<0>} \otimes c_{0<0><0>} \otimes c_{1<1>} \cdots c_{n<1>} c_{0<0><1>} \otimes mc_{0<1>} \\
&= c_{1<0>} \otimes \cdots \otimes c_{n<0>} \otimes c_{0<0>} \otimes c_{1<1>} \cdots c_{n<1>} c_{0<1>}^{(1)} \otimes mc_{0<1>}^{(2)} \\
&= c_{1<0>} \otimes \cdots \otimes c_{n<0>} \otimes c_{0<0>} \otimes c_{0<1>}^{(2)} c_{1<1>} \cdots c_{n<1>} \otimes mc_{0<1>}^{(1)} \\
&= c_1 \otimes \cdots \otimes c_n \otimes c_{0<0>} \otimes 1 \otimes mc_{0<1>}.
\end{aligned}$$

We use (1.13) in the second equality and $c_0 \otimes \cdots \otimes c_n \otimes m \in C^{\otimes(n+1)} \square_{\mathcal{H}} M$ in the last equality. □

Example 1.21. Let $\mathcal{H} = \mathcal{F} \blacktriangleright \mathcal{U}$ be a bicrossed product Hopf algebra where \mathcal{F} is commutative and cocommutative. The following computation proves that the coaction of \mathcal{H} on \mathcal{U} defined in (1.12) is cocommutative.

$$\begin{aligned}
& \tilde{u}_{<0>} \otimes d_{<0>} \otimes \tilde{u}_{<1>} d_{<1>}^{(1)} \otimes d_{<1>}^{(2)} \\
&= u_{1<0>} \otimes \cdots \otimes u_{n<0>} \otimes d_{<0>} \otimes u_{1<1>} \cdots u_{n<1>} d_{<1>}^{(1)} \otimes d_{<1>}^{(2)} \\
&= u_{1<0>} \otimes \cdots \otimes u_{n<0>} \otimes d_{<0>} \otimes (u_{1<1>} \blacktriangleright 1) \cdots (u_{n<1>} \blacktriangleright 1) (d_{<1>}^{(1)} \blacktriangleright 1) \otimes \\
&\quad \otimes (d_{<1>}^{(2)} \blacktriangleright 1) \\
&= u_{1<0>} \otimes \cdots \otimes u_{n<0>} \otimes d_{<0>} \otimes (u_{1<1>} \cdots u_{n<1>} d_{<1>}^{(1)} \blacktriangleright 1) \otimes (d_{<1>}^{(2)} \blacktriangleright 1) \\
&= u_{1<0>} \otimes \cdots \otimes u_{n<0>} \otimes d_{<0>} \otimes (d_{<1>}^{(2)} u_{1<1>} \cdots u_{n<1>} \blacktriangleright 1) \otimes (d_{<1>}^{(1)} \blacktriangleright 1) \\
&= u_{1<0>} \otimes \cdots \otimes u_{n<0>} \otimes d_{<0>} \otimes (d_{<1>}^{(2)} \blacktriangleright 1) (u_{1<1>} \blacktriangleright 1) \cdots (u_{n<1>} \blacktriangleright 1) \otimes \\
&\quad \otimes (d_{<1>}^{(1)} \blacktriangleright 1) \\
&= \tilde{u}_{<0>} \otimes d_{<0>} \otimes d_{<1>}^{(2)} \tilde{u}_{<1>} \otimes d_{<1>}^{(1)}.
\end{aligned}$$

In the fourth equality we use the commutativity and cocommutativity of \mathcal{F} . As an example, co-opposite Hopf algebra of Schwarzian Hopf algebra \mathcal{H}_{1s}^{cop} coacts cocommutatively on \mathcal{U} .

One has the following proper inclusions of categories.

$$SAYD_{\mathcal{H}} \subsetneq {}^C\mathcal{H}\text{-}SAYD \subsetneq {}^C\mathcal{H}\text{-}\mathcal{H}CC.$$

Remark 1.22. In order to consider all classes of well-defined coefficients for Hopf cyclic cohomology of a Hopf algebra and its generalizations, one should reconsider the related subjects regarding to the new understanding of the notion of coefficients of Hopf cyclic cohomology introduced in [HKR] and this paper.

The notion of SAYD contramodules which first introduced in [BR] can be generalized in a similar manner. As an example, for any \mathcal{H} -module algebra A , one can define the notion of ${}_A\mathcal{H}$ -SAYD contramodules. More precisely, Let \mathcal{H} be Hopf algebra, A a left \mathcal{H} module algebra. A left-right ${}_A\mathcal{H}$ -anti-Yetter-Drinfeld contramodule \mathcal{M} is a left \mathcal{H} module and a right \mathcal{H} -contramodule with the structure map $\alpha : \text{Hom}(\mathcal{H}, \mathcal{M}) \rightarrow \mathcal{M}$ such that for all $h \in \mathcal{H}$, $a' \in A$, $\tilde{a} \in A^{\otimes(n+1)}$ and $f \in \text{Hom}_{\mathcal{H}}(A^{\otimes(n+1)}, \mathcal{M})$ the following condition satisfies;

$$\alpha \left(h^{(2)} f \left(S(h^{(3)})(-) h^{(1)} a' \otimes \tilde{a} \right) \right) = h \alpha \left(f \left(S^{-1}(-) a' \otimes \tilde{a} \right) \right).$$

We say \mathcal{M} is stable if

$$\alpha(f((-) \triangleright \tilde{a})) = f(\tilde{a}).$$

Also one defines ${}_A\mathcal{H}$ -HCC contramodules as a modules where fit in the co-cyclic module [BR, page 4]. It can be shown that any SAYD contramodule over \mathcal{H} is an ${}_A\mathcal{H}$ -SAYD contramodule. Furthermore any ${}_A\mathcal{H}$ -SAYD contramodule is a ${}_A\mathcal{H}$ -HCC with respect to the related co-cyclic module defined in [BR].

One can also apply the approaches introduced in [HKR] and this paper to study suitable coefficients for Hopf cyclic cohomology of \times -Hopf algebras defined [BS], Hopf cyclic cohomology of \times -Hopf coalgebras introduced in [HR2], coefficients of cyclic cohomology of \times -Hopf algebras [KK], bialgebras [Ka] and Hopf algebras [S].

2 Pairing between module algebras and comodule algebras

The authors in [HKR] have generalized the pairing between Hopf cyclic cohomology of module algebras and module coalgebras which introduced in [KR, R1]. In this section we generalize the pairing between Hopf cyclic cohomology of module algebras and comodule algebras. Let \mathcal{H} be a Hopf algebra, A a left \mathcal{H} -module algebra, B a left \mathcal{H} -comodule algebra and \mathcal{M} a right-left ${}_A\mathcal{H}$ - and ${}^B\mathcal{H}$ -SAYD module. We consider the crossed product algebra $A \rtimes B$ with the following multiplication,

$$(a \rtimes b)(a' \rtimes b') = ab_{<-1>} a' \rtimes b_{<0>} b'. \quad (2.1)$$

This is an unital algebra where its unit element is $1 \rtimes 1$. Let $C^{\mathcal{H}}(A, M) := \text{Hom}^{\mathcal{H}}(A^{\otimes(n+1)}, M)$ and $C_{\mathcal{H}}^n(A, M) = \text{Hom}_{\mathcal{H}}(M \otimes A^{\otimes(n+1)}, \mathbb{C})$ be the co-cyclic modules defined in (1.3) and [HKRS2] respectively. We consider the following diagonal complex,

$$C_{a-a}^{n,n} := \text{Hom}_{\mathcal{H}}(M \otimes A^{\otimes(n+1)}, \mathbb{C}) \otimes \text{Hom}^{\mathcal{H}}(B^{\otimes(n+1)}, M) \quad (2.2)$$

with the cocyclic structure $(\delta \times d, \sigma \times s, \tau \times t)$. We define the following map,

$$\begin{aligned} \Psi : C_{a,a}^{n,n} &\longrightarrow \text{Hom}((A \rtimes B)^{\otimes(n+1)}, \mathbb{C}) \\ \Psi(\varphi \otimes \psi)(a_0 \rtimes b_0 \otimes \cdots \otimes a_n \rtimes b_n) &= \\ \varphi(\psi(b_{0<0>} \otimes \cdots \otimes b_{n<0>})) \otimes S^{-1}(b_{0<1>} \cdots b_{n<-1>}) a_0 \otimes \cdots \otimes S^{-1}(b_{n<-n-1>} a_n). \end{aligned}$$

Similar to the argument in [R1] one proves the following statement.

Lemma 2.1. *The map Ψ is a cocyclic map between cocyclic modules $C^{*,*}$ and $C^*(A \rtimes B)$.*

Therefore Ψ induces a map on the level of cyclic cohomology. One use the ${}^B\mathcal{H}$ -SAYD module property to use the similar argument in [R1] for proving the following proposition.

Proposition 2.2. *Let \mathcal{H} be a Hopf algebra, A a left \mathcal{H} -module algebra, B a left \mathcal{H} -comodule algebra and M a right-left ${}_A\mathcal{H}$ and ${}^B\mathcal{H}$ -SAYD module. The following map defines a cup product on Hopf cyclic cohomology.*

$$\begin{aligned} \sqcup : HC_{\mathcal{H}}^p(A, M) \otimes HC_{\mathcal{H}}^q(B, M) &\longrightarrow HC^{p+q}(A \rtimes B), \\ \sqcup &:= \Psi \circ AW. \end{aligned} \tag{2.3}$$

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